

POLYNOMIAL INVARIANTS OF QUANTUM CODES

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ABSTRACT. The weight enumerators ([4]) of a quantum code are quite powerful tools for exploring its structure. As the weight enumerators are quadratic invariants of the code, this suggests the consideration of higher-degree polynomial invariants. We show that the space of degree k invariants of a code of length n is spanned by a set of basic invariants in one-to-one correspondence with S_k^n . We then present a number of equations and inequalities in these invariants; in particular, we give a higher-order generalization of the shadow enumerator of a code, and prove that its coefficients are nonnegative. We also prove that the quartic invariants of a $((4, 4, 2))$ are uniquely determined, an important step in a proof that any $((4, 4, 2))$ is additive ([2]).

In [4], Shor and Laflamme introduced the concept of the weight enumerator of a quantum code, in order to prove a bound on the minimum distance of a code, given its length and dimension. The weight enumerators have the following two properties: equivalent codes have equal weight enumerators, and the coefficients of the weight enumerators are quadratic functions of the projection matrix associated to the code. More concisely, we can say that the coefficients of the weight enumerators are quadratic invariants of the code. In the present work, we will consider more general polynomial invariants.

The first task in the exploration is, naturally, to give a characterization of all polynomial invariants. Clearly, the invariants of any fixed degree form a vector space, so it suffices to give a set of invariants that span that space. This role is played by what we will call *basic* invariants; as we shall see, the basic invariants of order k and length n are in one-to-one correspondence with the group S_k^n . In the case of quadratic invariants, we recover the unitary weight enumerators of [3].

In [3], a conjecture is made regarding the *shadow enumerator* of a quantum code, in the case of alphabet size greater than 2. It turns out that this has a natural generalization to higher-order invariants; moreover, the structure of the generalization suggests a natural proof, thus settling that conjecture, and strengthening the linear programming bound for non-binary quantum codes.

The number of basic invariants is prohibitively large ($k!^n$) for n and k of any size. In order to render these invariants tractable, it is thus necessary that a number of linear dependences be found between them. In particular, it turns out that in a number of cases, an invariant of order k can be shown to be equal to an invariant of order $k - 1$. In addition, if the order of the invariant is greater than the alphabet

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size, we get further relations. In some cases, this reduces the degrees of freedom to the point of tractability.

We examine how these relations can be used in the case when the quantum code is a $((4, 4, 2))$; that is, when the code encodes 4 states in 4 qubits, with minimum distance 2. In this case, the available relations allow one to reduce the 331776 basic quartic invariants down to six degrees of freedom, which can be determined using more ad-hoc methods. In particular, we conclude that any two $((4, 4, 2))$ s must have the same quartic invariants. In [2], this fact is used to prove that any $((4, 4, 2))$ is equivalent to an additive code, and similarly for any $((5, 2, 3))$ or $((6, 1, 4))$, proving the uniqueness of each of those codes.

BASIC POLYNOMIAL INVARIANTS

For our purposes, it will be convenient to consider two types of polynomial invariants. Let Q be a quantum code of length n , dimension K , and alphabet size α ; let P_Q be the associated projection operator. A local polynomial invariant of Q is defined as a polynomial function f of the coefficients of P_Q such that

$$f(\phi P_Q \phi^{-1}) = f(P_Q)$$

for any $\phi \in U(\alpha)^{\otimes n}$. A global invariant is then defined as a local invariant that is also left unchanged under arbitrary permutations of the letters of Q . While global invariants are the only *true* invariants of the code, the structure of local invariants is simpler, and determines the global structure; we will therefore begin by considering local polynomial invariants.

Any polynomial function $f(M)$ in the coefficients of a matrix M can be written in the following form:

$$\sum_k \text{Tr}(F^{(k)} M^{\otimes k}),$$

for a suitable set of matrices $F^{(k)}$ on the domain of $M^{\otimes k}$. This can be seen by noting that a monomial of degree k in the coefficients of M can be written as

$$\prod_{1 \leq i \leq n} \text{Tr}(E_i M),$$

where each E_i has exactly one coefficient nonzero. But this is the same as

$$\text{Tr}((E_i)^{\otimes k} M^{\otimes k}).$$

Summing over monomials and over k , we get the desired expression. Now, consider how this expression changes when we conjugate M by a local equivalence ϕ :

$$\sum_k \text{Tr}(F^{(k)} \phi^{\otimes k} M^{\otimes k} (\phi^{\otimes k})^{-1}) = \sum_k \text{Tr}((\phi^{\otimes k})^{-1} F^{(k)} \phi^{\otimes k} M^{\otimes k}).$$

In particular, we can average over all local equivalences (since the group $U(\alpha)^{\otimes n}$ is compact) to obtain a polynomial invariant, and any polynomial invariant can be taken of that form. But this is equivalent to requiring that

$$(1) \quad F^{(k)} = (\phi^{\otimes k})^{-1} F^{(k)} \phi^{\otimes k}$$

for all $\phi \in U(\alpha)^{\otimes n}$. Thus we have reduced our classification problem to that of classifying the matrices $F^{(k)}$ satisfying (1).

Suppose, for a moment, that n is 1, so the group of local equivalences is the entire unitary group. In this case, the classical theory of invariants of the classical groups (see, for instance, [1]) tells us that the space of invariant $F^{(k)}$ s is spanned by a set of basic invariants in one-to-one correspondence with the symmetric group S_k . To be precise, for $\pi \in S_k$, the corresponding basic invariant is

$$T(\pi) = \delta_{i_{\pi(1)}}^{i_1} \delta_{i_{\pi(2)}}^{i_2} \delta_{i_{\pi(3)}}^{i_3} \dots \delta_{i_{\pi(k)}}^{i_k}.$$

Alternatively, if we consider $(\mathbb{C}^\alpha)^{\otimes k}$ as a k -letter Hilbert space, $T(\pi)$ is the operator which permutes the k qubits according to the permutation π . For $n > 1$, we can simply remark that the space of degree k invariants of a tensor product of groups is equal to the tensor product of the invariant spaces associated to each group individually. In particular, this gives us basic invariants in one-to-one correspondence with S_k^n . The corresponding operators $T(\pi)$ for $\pi \in S_k^n$ act on $((\mathbb{C}^\alpha)^{\otimes n})^{\otimes k}$ by permuting the k copies of the i th qubit according to the i th permutation in π .

Theorem 1. *Let $f(Q)$ be a polynomial invariant of quantum codes of length n and alphabet size α . Then there exists a sequence of functions f_k on S_k^n , eventually zero, such that*

$$f(Q) = \sum_{0 \leq k} \sum_{\pi \in S_k^n} f_k(\pi) \text{Tr}(T(\pi)(P_Q)^{\otimes k})$$

for all codes Q in the domain of f . If f is a global invariant, then $f_k(\pi)$ can be taken to be invariant under arbitrary permutations of the n subpermutations of π .

Proof. The above discussion has proved everything except for the comment on global invariants; this follows easily from considering the effect that reordering the qubits has on the basic invariants. \square

When $k = 2$, we recover the unitary weight enumerator of [3]. To be precise, note that S_2^n is in one-to-one correspondence with the set of subsets of $\{1, 2, \dots, n\}$; to $\pi \in S_2^n$, we associate the set $S(\pi)$ of indices such that π_i is a transposition. Then, using the notation of [3],

Theorem 2. *For any $\pi \in S_2^n$, and any Hermitian operators M and N ,*

$$\text{Tr}(T(\pi)(M \otimes N)) = A'_{S(\pi)}(M, N).$$

Proof. Recall that $A'_S(M, N)$ is defined by

$$A'_S(M, N) = \text{Tr}(\text{Tr}_{S^c}(M) \text{Tr}_{S^c}(N)).$$

Now, for all i in S^c , we readily see that

$$\text{Tr}(T(\pi)(M \otimes N)) = \text{Tr}(T(\pi^{(i)})(\text{Tr}_i(M) \otimes \text{Tr}_i(N))),$$

where $\pi^{(i)}$ is the tuple of permutations obtained from π by removing the i th permutation. This follows, for example, by noting that we can conjugate M by an

arbitrary unitary operation on the i th qubit without changing the result. But then we get

$$\mathrm{Tr}(T(\pi)(M \otimes N)) = \mathrm{Tr}(T((12))(\mathrm{Tr}_{S^c}(M) \otimes \mathrm{Tr}_{S^c}(N))) = A'_S(M, N).$$

□

This motivates the notation

$$A'_\pi(Q) = \mathrm{Tr}(T(\pi)P_Q^{\otimes k}).$$

Similarly, if we are given a k -tuple of Hermitian matrices, we write

$$A'_\pi(M_1, M_2, \dots, M_k) = \mathrm{Tr}(T(\pi)(M_1 \otimes M_2 \otimes \dots \otimes M_k)).$$

GENERALIZED SHADOW INEQUALITIES

In [3], the author made the following conjecture:

Conjecture. *Let $V = V_1 \otimes V_2 \otimes \dots \otimes V_n$, where V_1 through V_n are Hilbert spaces. Let T be any subset of $\{1, 2, \dots, n\}$, and let M and N be positive semi-definite Hermitian operators on V . Then*

$$\sum_{S \subset \{1, 2, \dots, n\}} (-1)^{|S \cap T|} \mathrm{Tr}(\mathrm{Tr}_{S^c}(M) \mathrm{Tr}_{S^c}(N)) \geq 0.$$

In particular, when $M = N = P_Q$, this gives an inequality that the quadratic invariants of a code must satisfy. It turns out that this generalizes naturally to higher order invariants. Let us first restate this conjecture in terms of basic quadratic invariants:

$$\begin{aligned} \kappa_T(M, N) &= \sum_{\pi \in S_2^n} (-1)^{|S(\pi) \cap T|} A'_\pi(M, N) \\ &= \sum_{\pi \in S_2^n} \lambda_T(\pi) A'_\pi(M, N). \\ &\geq 0, \end{aligned}$$

where

$$\lambda_T(\pi) = (-1)^{|S(\pi) \cap T|}.$$

Note that $\lambda_T(\pi)$ is a Hermitian idempotent in the group algebra of S_2^n ; this suggests the following generalization:

Theorem 3. *Let $V = V_1 \otimes V_2 \otimes \dots \otimes V_n$, where V_1 through V_n are Hilbert spaces. Let T be any subset of $\{1, 2, \dots, n\}$, and let M_1, M_2, \dots, M_k be positive semi-definite Hermitian operators on V . Then for any Hermitian idempotent λ in the group algebra of S_k^n ,*

$$\sum_{\pi \in S_k^n} \lambda(\pi) A'_\pi(M_1, M_2, \dots, M_k) \geq 0.$$

Proof. The given expression is multilinear in the M_i ; consequently, it suffices to consider the case in which each M_i has rank one. Thus, let v_i for each i be a vector in V such that $M_i = v_i v_i^\dagger$. This allows us to restate our question as showing that:

$$\kappa = \sum_{\pi \in S_k^n} \lambda(\pi) A'_\pi(v_i v_i^\dagger) \geq 0.$$

Rewriting A'_π as a trace, we have:

$$\begin{aligned} \kappa &= \sum_{\pi \in S_k^n} \lambda(\pi) (v_1 \otimes v_2 \otimes \dots \otimes v_k)^\dagger T(\pi) (v_1 \otimes v_2 \otimes \dots \otimes v_k) \\ &= \sum_{\pi, \pi' \in S_k^n} \lambda(\pi \pi'^{-1}) \lambda(\pi') (v_1 \otimes v_2 \otimes \dots \otimes v_k)^\dagger T(\pi) (v_1 \otimes v_2 \otimes \dots \otimes v_k) \\ &= \sum_{\pi_1, \pi_2 \in S_k^n} \lambda(\pi_1) \lambda(\pi_2) (v_1 \otimes v_2 \otimes \dots \otimes v_k)^\dagger T(\pi_1) T(\pi_2) (v_1 \otimes v_2 \otimes \dots \otimes v_k) \\ &= \left(\sum_{\pi_1 \in S_k^n} \lambda(\pi_1) T(\pi_1) v_1 \otimes v_2 \otimes \dots \otimes v_k \right)^\dagger \left(\sum_{\pi_2 \in S_k^n} \lambda(\pi_2) T(\pi_2) v_1 \otimes v_2 \otimes \dots \otimes v_k \right). \end{aligned}$$

But this is the norm of a vector, so must be nonnegative. \square

By the discussion preceding the theorem, the above conjecture follows as an immediate corollary.

REDUCTIONS AND RELATIONS

Although we have shown that the basic invariants span the space of polynomial invariants, we have by no means shown that they form a basis. Indeed, there are a number of linear equations relating the various basic invariants. For example, if we conjugate every permutation in π by a fixed element of S_k , the corresponding basic invariant will be unchanged; this corresponds to the fact that the different copies of P_Q appearing in the expression for the basic invariant can be freely interchanged.

Many of these equations take the form of a reduction, in which an invariant of degree k is expressed as an invariant of degree $k - 1$. The most general of these reductions follows from the fact that P_Q is a projection operator, so $P_Q^2 = P_Q$. Suppose there is an index $1 \leq j \leq n$ such that $\pi_i(j)$ is constant as i ranges from 1 to k . Then each qubit of the j th copy of P_Q is connected to the corresponding qubit of the $\pi_i(j)$ th copy. But then this gives us P_Q^2 , which we can replace by P_Q .

For example, consider the invariant

$$(2) \quad A'_{(12)(345), (123)(45), (124)(35), (125)(34)}(P_Q).$$

Here, each permutation maps 1 to 2; in consequence, we can merge 1 and 2, obtaining:

$$A'_{(1)(345), (13)(45), (14)(35), (15)(34)}(P_Q),$$

which we can renumber as

$$A'_{(1)(234), (12)(34), (13)(24), (14)(23)}(P_Q),$$

which we are unable to reduce further.

In some cases, the information we are given concerning Q allows us to give further reductions. For example, suppose we are given a set S such that $A'_S(Q) = 2^{-|S|}A'_\emptyset(Q)$; if Q is known to be pure to weight w , for instance, then this holds for all S of size less than w . Then $\text{Tr}_{S^c}(P_Q)$ is proportional to an identity matrix (with some known constant depending only on the dimension of Q). Suppose there exists a j with $1 \leq j \leq k$ such that $\pi_i(j) = j$ for all i in S^c . Then we can splice j out of each permutation, resulting in a lower-order invariant that is a constant multiple of the original invariant.

For instance, suppose that in the above example we knew that $\text{Tr}_{S^c}(P_Q) = \frac{1}{2}I$ for all S of cardinality 3; this is the case when Q is a binary $((4, 4, 2))$, for instance. Then we can reduce

$$A'_{(1)(234),(12)(34),(13)(24),(14)(23)}(P_Q)$$

by splicing out 1, obtaining

$$\frac{1}{2}A'_{(234),(2)(34),(3)(24),(4)(23)}(P_Q),$$

which reduces further to

$$\frac{1}{4}A'_{(34),(34),(3)(4),(3)(4)}(P_Q) = \frac{1}{4}A'_{\{1,2\}}(P_Q).$$

Thus we have reduced the quintic invariant (2) to a quadratic invariant. It should be apparent, therefore, that these reductions can be a powerful tool in simplifying higher-order invariants.

A final important class of relations appears when the order of the invariant is greater than the alphabet size. For $n = 1$, we have the following classical result:

Lemma 4. *If $k > \alpha$, then*

$$\sum_{\pi \in S_k} \sigma(\pi)T(\pi) = 0.$$

Proof. Let M be a matrix of dimension k . Then one readily sees that

$$\text{Tr}\left(\left(\sum_{\pi \in S_k} \sigma(\pi)T(\pi)\right)M^{\otimes k}\right) = \det(M),$$

from the definition of determinant. Now, the basic invariants are unchanged if we enlarge each matrix by adding a row and column of zeros. Consequently, for M of dimension less than k (i.e., α),

$$\text{Tr}\left(\left(\sum_{\pi \in S_k} \sigma(\pi)T(\pi)\right)M^{\otimes k}\right) = 0.$$

The only way this can happen for all M is if

$$\sum_{\pi \in S_k} \sigma(\pi)T(\pi) = 0.$$

□

We get further relations by adding fixed points, or multiplying the sum of T s by some fixed T . For instance, consider the invariant

$$A'_{(123),(123),(132)}(Q),$$

in the case when Q is a binary code. Then lemma 4 tells us that

$$\begin{aligned} A'_{(123),(123),(132)}(Q) + A'_{(123),(123),(1)(2)(3)}(Q) + A'_{(123),(123),(123)}(Q) \\ = A'_{(123),(123),(12)(3)}(Q) + A'_{(123),(123),(13)(2)}(Q) + A'_{(123),(123),(23)(1)}(Q). \end{aligned}$$

The last four terms always reduce to quadratic invariants, while the second term sometimes admits reduction as well. We also have, for example,

$$\begin{aligned} A'_{(12)(34),(123)(4)}(Q) + A'_{(134)(2),(123)(4)}(Q) + A'_{(234)(1),(123)(4)}(Q) \\ = A'_{(1)(2)(34),(123)(4)}(Q) + A'_{(1342),(123)(4)}(Q) + A'_{(1234),(123)(4)}(Q), \end{aligned}$$

obtained by multiplying the relation

$$\begin{aligned} T((1)(2)(3)(4)) + T((123)(4)) + T((132)(4)) \\ = T((12)(3)(4)) + T((13)(2)(4)) + T((1)(23)(4)) \end{aligned}$$

by $T((12)(34))$. As we shall see in the next section, these relations derived from lemma 4 can be extremely powerful.

BINARY MDS CODES OF DISTANCE 2

Let us consider the case when Q is a binary MDS code of distance 2; that is, when Q is a $((2m, 4^{m-1}, 2))$. In this case, $\text{Tr}_i(P_Q) = \frac{1}{2}I$ for $1 \leq i \leq 2m$. This allows us to reduce any basic invariant containing a non-derangement (a permutation with a fixed point) to a lower-order invariant, as we have just seen.

Lemma 5. *For each $m \geq 2$, the local cubic invariants of a $((2m, 4^{m-1}, 2))$ are uniquely determined. That is, if Q and Q' are $((2m, 4^{m-1}, 2))$ s, and $\pi \in S_3^{2m}$, then $A'_\pi(Q) = A'_\pi(Q')$.*

Proof. Since a $((2m, 4^{m-1}, 2))$ is MDS, its local quadratic invariants are uniquely determined. Therefore, it suffices to restrict our attention to those invariants corresponding to π consisting entirely of derangements; otherwise, the invariant can be reduced to a local quadratic invariant, and is thus uniquely determined.

Since we are dealing with a binary code, lemma 4 applies:

$$T((1, 3, 2)) = -T((1)(2)(3)) - T((1, 2, 3)) + T((1, 2)(3)) + T((1, 3)(2)) + T((2, 3)(1)).$$

In particular, this allows us to reduce any invariant involving $(1, 3, 2)$ to a sum of invariants involving only $(1, 2, 3)$ or permutations with fixed points. Thus the only remaining non-derangement invariant is $A'_{(1,2,3)^{2m}}(Q)$, which reduces to $\text{Tr}(P_Q^3) = 2^{m-2}$. □

For a $((4, 4, 2))$, we can say more:

Theorem 6. *The local quartic invariants of a $((4, 4, 2))$ are uniquely determined.*

Proof. Let Q be a $((4, 4, 2))$. As in lemma 5, we may restrict our attention to derangements. For convenience, we define

$$\pi_1 = (1, 2, 3, 4), \quad \pi_2 = (1, 3, 4, 2), \quad \pi_3 = (1, 4, 2, 3).$$

Then any derangement in S_4 can be written as π_i^j for $i, j \in \{1, 2, 3\}$. Furthermore, we have the following relations in the representation T , from lemma 4:

$$(3) \quad \begin{aligned} \pi_1^3 &= \pi_1 + \text{n.d.}, & \pi_2^3 &= \pi_2 + \text{n.d.}, & \pi_3^3 &= \pi_3 + \text{n.d.} \\ \pi_1^2 &= \pi_2 + \pi_3 + \text{n.d.}, & \pi_2^2 &= \pi_1 + \pi_3 + \text{n.d.}, & \pi_3^2 &= \pi_1 + \pi_2 + \text{n.d.}, \end{aligned}$$

where n.d. refers to some linear combination of non-derangements. This allows us to restrict our attention to invariants involving only π_1 , π_2 , and π_3 . Now, note that $\pi_1(3) = \pi_2(3)$, $\pi_2(4) = \pi_3(4)$, and $\pi_1(2) = \pi_3(2)$. It follows that any invariant involving only two of the three can be reduced to a cubic invariant. We therefore have only six degrees of freedom remaining, corresponding to the local invariant

$$A'_{\pi_1, \pi_1, \pi_2, \pi_3}(Q),$$

and its six permutations. In particular, we have only one degree of freedom remaining in the global invariants.

Now, let v be any codeword in Q , and consider $\text{Tr}_{\{234\}}(vv^\dagger)$. Since Q is pure to distance 2, it follows that

$$\text{Tr}_{\{234\}}(vv^\dagger) = 2I.$$

Thus the commutator

$$[\text{Tr}_{\{234\}}(vv^\dagger) \otimes I, P_Q] = 0$$

for all $v \in Q$. But then

$$E_{v \in Q}(\text{Tr}(-[\text{Tr}_{\{234\}}(vv^\dagger) \otimes I, P_Q]^2)) = 0,$$

where v may be taken to be normally distributed. This is a local quartic invariant of Q , equal to

$$\begin{aligned} & 2A'_{(1,2,3,4),(1)(2)(3,4),(1)(2)(3,4),(1)(2)(3,4)}(Q) \\ & + 2A'_{(1)(2,3,4),(1,2)(3,4),(1,2)(3,4),(1,2)(3,4)}(Q) \\ & - 2A'_{(1,3,2,4),(1)(2)(3,4),(1)(2)(3,4),(1)(2)(3,4)}(Q) \\ & - 2A'_{(1,3)(2,4),(1,2)(3,4),(1,2)(3,4),(1,2)(3,4)}(Q). \end{aligned}$$

Simplifying along fixed points, we conclude that

$$A'_{(1,3)(2,4),(1,2)(3,4),(1,2)(3,4),(1,2)(3,4)}(Q) = A'_{\pi_1^2, \pi_3^2, \pi_3^2, \pi_3^2} = 4.$$

Applying the reductions (3) to the first qubit, we find

$$A'_{\pi_3, \pi_3^2, \pi_3^2, \pi_3^2}(Q) = 2,$$

then, reducing the second through fourth qubits,

$$A'_{\pi_3, \pi_1, \pi_3^2, \pi_3^2}(Q) + A'_{\pi_3, \pi_2, \pi_3^2, \pi_3^2}(Q) = 2,$$

$$A'_{\pi_3, \pi_1, \pi_1, \pi_3^2}(Q) + A'_{\pi_3, \pi_2, \pi_1, \pi_3^2}(Q) + A'_{\pi_3, \pi_1, \pi_2, \pi_3^2}(Q) + A'_{\pi_3, \pi_2, \pi_2, \pi_3^2}(Q) = 2,$$

and

$$A'_{\pi_3, \pi_1, \pi_1, \pi_2} + A'_{\pi_3, \pi_1, \pi_2, \pi_1} + A'_{\pi_3, \pi_2, \pi_1, \pi_1} = 3.$$

Permuting this equation gives us four more equations relating the local invariants, leaving two degrees of freedom. However, this is enough to determine the global invariants:

$$(A'_{\pi_1, \pi_1, \pi_2, \pi_3})_{\text{sym}} = 1.$$

Now, consider, for $S \subset \{1, 2, 3, 4\}$ of size 2,

$$\kappa_S = E_{v \in Q}(\text{Tr}(-[\text{Tr}_{S^c}(vv^\dagger) \otimes I, P_Q]^2)).$$

This is a local quartic invariant, and further must be nonnegative, since the commutator of two Hermitian operators is anti-Hermitian. On the other hand, we have, for example,

$$\begin{aligned} \kappa_{\{1,2\}} \propto & 8 - 2A'_{\pi_2, \pi_3, \pi_1, \pi_1}(Q) - 2A'_{\pi_1, \pi_1, \pi_2, \pi_3}(Q) \\ & - A'_{\pi_1, \pi_2, \pi_1, \pi_3}(Q) - A'_{\pi_1, \pi_2, \pi_3, \pi_1}(Q) - A'_{\pi_2, \pi_1, \pi_1, \pi_3}(Q) - A'_{\pi_2, \pi_1, \pi_3, \pi_1}(Q). \end{aligned}$$

Symmetrizing, we find

$$\sum_S \kappa_S = 0.$$

But then the nonnegativity of the κ_S implies $\kappa_S = 0$ for each S . This gives us three further equations on the local invariants, eliminating the two remaining degrees of freedom. \square

FURTHER DIRECTIONS

It must be stressed that the relations and inequalities we have derived above by no means exhaust the possibilities; for instance, it should be possible to define a higher-order, but still nonnegative, analogue of the A_S weight enumerators, which would produce a number of inequalities on the higher-order invariants. Furthermore, it seems clear that we are not taking full advantage of the relations that can be deduced from the minimum distance of the code. Progress needs to be made in these areas in order for polynomial invariants to be truly useful.

In addition, it should be noted in passing that there is some evidence (see the remark after corollary 9 in [2]) that some simple set of relations on the quartic invariants, satisfied by all additive codes, are enough to force a code to be additive. This possibility probably merits further investigation.

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